W2/2 Free and projective modules

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Exm: If $I \trianglelefteq A$, $M \in A\text{-Mod}$, then $M/_{IM} \cong M \otimes_A A/_I$.

[$0 \to I \to A \to A/_I \to 0$ is a SES of A-modules.

$\Rightarrow$ $I \otimes M \longrightarrow A \otimes M \longrightarrow A/_I \otimes M \to 0$ is exact.

Now $\mu: A \otimes_A M \to M$, $a \otimes m \mapsto am$ is an iso (inverse: $m \mapsto 1 \otimes m$),
with $\mu(I \otimes M) = IM \Rightarrow A/_I \otimes M \cong M/_{IM}$.]

Prop: If $A \neq 0$ and $A^{(I)} \cong A^{(J)}$ for sets I, J, then $|I| = |J|$.

Proof: Let $M \in \mathrm{Max}(A)$. Then $A/_M$ is a field K and

$$A^{(I)} \otimes_A A/_M \cong (A \otimes_A A/_M)^{(I)} \cong K^{(I)}$$ as A-modules and

also as $K = A/_M$-vector spaces. So $A^{(I)} \cong A^{(J)}$ implies

$K^{(I)} \cong K^{(J)}$, and thus $|I| = |J|$, bec. the dimension of a K-vector space is well-defined. $\square$

Def: (For $A \neq 0$) If M is a f.g. free module, its rank is the (unique) $n \in \mathbb{N}_0$ s.t. $M \cong A^n_A$

1.3 Projective, Injective, Flat Modules

Let $M \in A\text{-Mod}$.

$M \otimes -$ is i.g. not left exact, e.g.

$(*)$ $0 \to 2\mathbb{Z} \hookrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/_{2\mathbb{Z}} \to 0$,

but the induced $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_{2\mathbb{Z}} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_{2\mathbb{Z}}$ is not injective but the induced $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is <u>not</u> injective

$\mathrm{Hom}(M,-)$, $\mathrm{Hom}(-,M)$ are <u>not</u> right exact i.g., e.g.

$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ applied to $(*)$:

$$0 \to \underset{\{0\}}{\underset{\parallel}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, 2\mathbb{Z})}} \to \underset{\{0\}}{\underset{\parallel}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})}} \to \underset{\{0, \mathrm{id}\}}{\underset{\parallel}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})}}$$

$\mathrm{Hom}(-,\mathbb{Z})$ applied to $(*)$:

$$\varphi \longmapsto \varphi \circ (2\mathbb{Z} \hookrightarrow \mathbb{Z}) = \varphi|_{2\mathbb{Z}}$$

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathrm{Hom}(2\mathbb{Z}, \mathbb{Z})$$

$\rotatebox{90}{$\in$}$ $2n \mapsto n$ is not in image

<u>Def</u>: A module M_A is

- ·) **projective** if $\mathrm{Hom}(M,-)$ is exact.
- ·) **injective** if $\mathrm{Hom}(-,M)$ is exact.
- ·) **flat** if $M \otimes -$ is exact.

<u>Thm 1.11</u> For $P \in A\text{-Mod}$, TFAE:

(a) P is projective

(b) For every epi $g: M_A \to N_A$, $\mathrm{Hom}(P,M) \xrightarrow{g_*} \mathrm{Hom}(P,N)$ is an epi.

(c) For every diagram w. exact row

$$\begin{tikzcd} & P \arrow[d, "\varphi"] \arrow[dl, dashed, "\exists \psi"'] & \\ M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

$\exists \psi \in \mathrm{Hom}(P,M)$ making it commutative $(\varphi = f \circ \psi)$.

(d) Every epi $g: M \to P$ splits (i.e., $\exists s: P \to M$ s.t. $g \circ s = \mathrm{id}_P$)

(e) There exists $M \in A\text{-Mod}$ s.t. $M \oplus P$ is free.

(d) Every epi $g: N \to P$ splits (i.e., $\exists s: P \to N$ s.t. $g \circ s = id_P$)

(e) There exists $M \in A$-Mod s.t. $M \oplus P$ is free.

<u>Proof:</u> (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c) $\quad f_*: \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$, $\alpha \mapsto f\circ\alpha$ is epi

$\Rightarrow \exists \psi \in \mathrm{Hom}(P,M): \ \varphi = f\circ\psi.$

(c) $\Rightarrow$ (d) Consider

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists s"'] \arrow[d, "id"] & \\ M \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

By (c) $\exists s \in \mathrm{Hom}(P,M)$ s.t. $g\circ s = id_P$.

(d) $\Rightarrow$ (e) Let $g: A^{(I)} \to P$ be an epi for some set I.

$\overset{(d)}{\Longrightarrow}$ $0 \to \ker(g) \hookrightarrow A^{(I)} \xrightarrow{g} P \to 0$ is a split SES

$\overset{L1.7}{\Longrightarrow}$ $A^{(I)} \cong P \oplus \ker(g)$

(e) $\Rightarrow$ (a) Let $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ be exact. Consider

$$0 \to \mathrm{Hom}(P,M) \xrightarrow{f_*} \mathrm{Hom}(P,N) \xrightarrow{g_*} \mathrm{Hom}(P,Q) \to 0$$

By left exactness of $\mathrm{Hom}(P,-)$, we only have to check surjectivity of g_* (i.e. (b))

Fix C, I s.t. $P \oplus C \cong A^{(I)}$, $\pi: A^{(I)} \to P$ canonical projection,
$\varepsilon: P \hookrightarrow A^{(I)}$ embedding ($\pi\circ\varepsilon = id_P$)

Let $\varphi \in \mathrm{Hom}(P,Q)$.

$$\begin{tikzcd} & A^{(I)} \arrow[ddl, dashed, "\Psi"'] \arrow[d, shift left, "\pi"] & \\ & P \arrow[u, shift left, "\varepsilon"] \arrow[d, "\varphi"] & \\ N \arrow[r, "g"] & Q \arrow[r] & 0 \end{tikzcd}$$

$\in A^{(I)}$ choose $n_i \in N$

$$N \xrightarrow{g} Q \longrightarrow 0$$

For each standard basis vector $e_i \in A^{(I)}$, choose $n_i \in N$ s.t. $g(n_i) = \varphi \circ \pi(e_i)$.

$\Rightarrow \exists \Psi \in \mathrm{Hom}(A^{(I)}, N)$; $\forall i \in I$: $\Psi(e_i) = n_i$ $\Rightarrow g \circ \Psi = \varphi \circ \pi$.

Let $\psi := \Psi|_P = \Psi \circ \varepsilon$

$\Rightarrow \forall p \in P$: $\underline{g \circ \psi(p)} = g \circ \Psi \circ \varepsilon(p) = \varphi \circ \underbrace{\pi \circ \varepsilon}_{\mathrm{id}_P}(p) = \underline{\varphi(p)}$

<u>so</u>: $g_*(\psi) = \varphi$. $\square$

<u>Exm</u>: Free modules are projective.

For $A = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is projective non-free